

The Seiberg-Witten map and supersymmetry

C.P. Martín and C. Tamarit

*Departamento de Física Teórica I, Facultad de Ciencias Físicas,
Universidad Complutense de Madrid,
28040 Madrid, Spain
E-mail: carmelo@elbereth.fis.ucm.es, ctamarit@fis.ucm.es*

ABSTRACT: The lack of any local solution to the first-order-in- $\hbar\omega^{mn}$ Seiberg-Witten (SW) map equations for U(1) vector superfields compels us to obtain the most general solution to those equations that is a quadratic polynomial in the ordinary vector superfield, v , its chiral and antichiral projections and the susy covariant derivatives of them all. Furnished with this solution, which is local in the susy Landau gauge, we construct an ordinary dual of noncommutative U(1) SYM in terms of ordinary fields which carry a linear representation of the $\mathcal{N} = 1$ susy algebra. By using the standard SW map for the $\mathcal{N} = 1$ U(1) gauge supermultiplet we define an ordinary U(1) gauge theory which is dual to noncommutative U(1) SYM in the WZ gauge. We show that the ordinary dual so obtained is supersymmetric, for, as we prove as we go along, the ordinary gauge and fermion fields that we use to define it carry a nonlinear representation of the $\mathcal{N} = 1$ susy algebra. We finally show that the two ordinary duals of noncommutative U(1) SYM introduced above are actually the same $\mathcal{N} = 1$ susy gauge theory. We also show in this paper that the standard SW map is never the $\theta\bar{\theta}$ component of a local superfield in v and check that, at least at a given approximation, a suitable field redefinition of that map makes the noncommutative and ordinary — in a B_{mn} field — susy U(1) DBI actions equivalent.

KEYWORDS: Supersymmetric gauge theory, D-branes, Non-Commutative Geometry.

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1. Introduction

Noncommutative quantum field theories have been widely investigated in the past years, chiefly after it was shown in ref. [1] that they arise as effective theories of open strings ending on D-Branes with a constant Neveu-Schwarz background B_{mn} . In ref. [1], it was also shown that noncommutative $U(1)$ gauge theories can be mapped to a theory with ordinary gauge symmetry, since both theories arise as effective theories of the same underlying open string theory; this equivalence can be seen [1] as a mapping between a noncommutative Moyal deformed DBI action and a commutative DBI action in the presence of a constant B_{mn} background. The Seiberg-Witten map thus associates to every noncommutative $U(N)$ gauge theory an equivalent — at least for energies well below the noncommutative energy scale — ordinary $U(N)$ gauge theory, which we shall call in the sequel the ordinary dual under the Seiberg-Witten map of the former noncommutative gauge theory.

Most of the papers — see refs. [2–12] and [13] for an incomplete list — where the properties of the ordinary duals under the Seiberg-Witten map of noncommutative $U(N)$ gauge theories are discussed deal with nonsupersymmetric theories or with the bosonic sector of supersymmetric theories. The construction of supersymmetric duals under the Seiberg-Witten map of noncommutative supersymmetric $U(N)$ gauge theories is tackled

only in an astonishingly short number of papers — see for instance refs. [14–20]. Moreover the picture emerging from them is a bit blurred since there are important issues that have not been clarified in them and which we shall spell out next. First, there is the issue of the existence of a generalisation to objects made out of superfields of the Seiberg-Witten map introduced in ref. [1] — the map in ref. [1] will be called henceforth the standard Seiberg-Witten map. In refs. [14] and [18] it is claimed that there exists such a generalisation and that it is a polynomial — and thus a local object — in the ordinary vector superfield and its supersymmetry covariant derivatives. This statement is at odds with the result presented in ref. [19] where it is shown that the first-order consistency condition for the Seiberg-Witten map for superfields admits no solution that is a polynomial of the appropriate ordinary superfields and their supersymmetry covariant derivatives. The latter result is in line with the fact that no local solution to the Seiberg-Witten map equations was found in ref. [15] at first order in the noncommutativity parameters and with the claim made in ref. [17] that there is no superfield formalism in terms of ordinary vector superfields that would allow us to formulate the ordinary dual under the standard Seiberg-Witten map of noncommutative $U(1)$ superYang-Mills theory in the Wess-Zumino gauge. In ref. [15] a solution to the Seiberg-Witten map equations for $U(1)$ superfields was worked out at first order in the noncommutativity parameters. In ref. [15], it is also claimed that the solution displayed in there is unique, which is quite surprising. The Seiberg-Witten map obtained by the authors of ref. [15] is local and trivial in the supersymmetric Landau gauge — but nonlocal and non-trivial otherwise — and yields an ordinary dual with linearly realised supersymmetry of the noncommutative $U(1)$ $\mathcal{N} = 1$ superYang-Mills theory. Secondly, there is the issue of the supersymmetric character of the ordinary dual under the standard Seiberg-Witten map of $U(1)$ superYang-Mills theory in the Wess-Zumino gauge. Such ordinary dual theory is constructed in refs. [17, 18] and [20]. In these papers, the transformations of the fields of the dual ordinary theory that give rise to the supersymmetry transformations of the noncommutative theory are computed at first-order in the noncommutativity parameters. Those transformations of the dual ordinary fields turn out to be nonlinear, though local, in these fields. It is thus apparent that these ordinary dual fields do not carry a linear realisation of the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions. Whether these nonlinear transformations constitute a nonlinear realisation of the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions is not discussed in those papers, although the transformations in question are referred to as supersymmetry transformations. We believe that to rightly call these transformations supersymmetry transformations one should establish first that they are nonlinear realisations of the supersymmetry algebra. It should also be noticed that in general the Seiberg-Witten map does not preserve the gauge-fixing condition — e.g., it does not map in general an ordinary gauge field configuration in the temporal gauge into a noncommutative gauge field configuration in that very gauge, a situation that is reproduced for the Wess-Zumino gauge for the superfield Seiberg-Witten map of ref. [15] — so it is not obvious that by choosing the Wess-Zumino gauge and then applying the standard Seiberg-Witten map one does not give rise to a breaking of supersymmetry in the ordinary dual theory so constructed. Now that there seem to arise two ordinary duals — one obtained by using a nonlocal superfield Seiberg-Witten map and

the other constructed by using the standard Seiberg-Witten map — of noncommutative $U(1)$ $\mathcal{N} = 1$ superYang-Mills theory, it is fair to ask whether they really are different theories as ordinary theories — they seem to have different supersymmetric features — or the same ordinary theory expressed in terms of different sets of field variables. We have just stated the third issue that has not been clarified yet. Let us mention that in gaining a full understanding of all these matters one should check — a check that has not been done in the literature yet — that the standard Seiberg-Witten map, or some Seiberg-Witten map equivalent to it, establishes a connection between the ordinary DBI action in the presence of a constant background B_{mn} field and the noncommutative DBI action for $\mathcal{N} = 1$ supersymmetry in four dimensions.

The purpose of this paper is to clarify all the issues commented upon above. Before we display how we have organised the paper, let us point out that a complete understanding of the duality relationship established by the Seiberg-Witten map for noncommutative supersymmetric gauge theories at the classical level is necessary, if the existence of such duality relationship is to be investigated for quantum theories — only for Chern-Simons theory such investigation has been undertaken [21]. Indeed, on the one hand, due to UV/IR mixing, noncommutative non-supersymmetric Yang-Mills theories have severe noncommutative infrared divergences that are absent in their supersymmetric versions [22], and, on the other hand, the ordinary dual theory of a given noncommutative gauge theory — e.g., noncommutative QED — is not necessarily renormalisable [23]. We would also like to stress that the results we shall report on below are also relevant to the field of noncommutative gauge theories constructed within the enveloping-algebra formalism. This formalism was put forward in refs. [24, 25] and [26], and has led to important new results such as the formulation of the noncommutative Standard Model [27] and other [28] anomaly free theories [29], which may be of relevance in accounting for the experimental data to be recorded at the LHC [30–33].

The layout of this paper is as follows. In section 2 we show by explicit computation that the standard Seiberg-Witten map is never the $\theta\bar{\theta}$ component of a superfield made out of the ordinary vector superfield and its supersymmetry covariant derivatives, and address the problem of finding physically sensible solutions to the Seiberg-Witten map equation for $U(1)$ superfields. Here, we construct, at first order in the noncommutativity parameters, the most general solution — which is not unique — to this equation that is a quadratic polynomial in the ordinary vector superfield, its chiral and antichiral projections and their supersymmetry covariant derivatives. We show in section 3 that the standard Seiberg-Witten map, applied to the noncommutative gauge supermultiplet of noncommutative $U(N)$ superYang-Mills theory in the Wess-Zumino gauge, always yields an ordinary $U(N)$ gauge supermultiplet which carries a nonlinear representation of the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions. We discuss here how this result is in agreement with the fact that, upon adding certain field redefinitions — that we compute in appendix B — the standard Seiberg-Witten map turns, in some approximation, the $\mathcal{N} = 1$ supersymmetric DBI action in the presence of a B_{mn} field into the noncommutative $\mathcal{N} = 1$ supersymmetric DBI action. In section 4 we show that the dual ordinary theories of noncommutative $U(1)$ superYang-Mills theories constructed in sections 1 and 2 are the same supersymmetric

theory but formulated in terms of different sets of variables. Our summary of the paper and the conclusions are the content of section 5. We also include three appendices. Appendix A is merely notational. In appendix B we discuss the equivalence under the Seiberg-Witten map of the DBI action in the presence of a B_{mn} field and the noncommutative DBI action, in the case of $\mathcal{N} = 1$ supersymmetry in four dimensions. We have included in appendix C the proof that the ordinary dual under the standard Seiberg-Witten map of noncommutative U(1) Yang-Mills theory cannot be turned into a supersymmetric theory by including in the action new local terms of the appropriate dimension, if the fields in the resulting action carry a linear representation of $\mathcal{N} = 1$ supersymmetry in four dimensions.

2. The Seiberg-Witten map equation for superfields and an ordinary dual of noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills

The aim of this section is to obtain a U(1) ordinary theory with linearly realised $\mathcal{N} = 1$ supersymmetry which is dual, at least classically, to noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills. To do so we shall set up the Seiberg-Witten-map equations for U(1) superfields and then build solutions to them. We shall also show that these solutions cannot be constructed by following the strategy suggested in ref. [14].

We define noncommutative gauge theories with linearly realised $\mathcal{N} = 1$ supersymmetry in terms of superfields as in refs. [34, 35]. Our superspace conventions will be those found in ref. [36] and the Moyal product, “ \star ”, of a and b will be given by $a \star b = a \exp\left(\frac{i\hbar}{2} \overleftarrow{\partial}_m \omega^{mn} \overrightarrow{\partial}_n\right) b$; \hbar sets the noncommutative scale. All along this paper, we will denote space-time indices with Latin letters and spinor indices with Greek letters. V shall denote a U(1) noncommutative vector superfield. Under noncommutative U(1) transformations — defined by the chiral superfield Λ — V transforms as follows:

$$e_{\star}^{V'} = e_{\star}^{i\bar{\Lambda}} \star e_{\star}^V \star e_{\star}^{-i\Lambda}, \tag{2.1}$$

e_{\star}^A denotes the exponential of A defined in terms of the usual power series with products replaced by star products. $\bar{\Lambda}$ is the conjugate of Λ .

Let s_{nc} denote the operator generating the noncommutative BRS transformations of the superfields V , then, eq. (2.1) leads to

$$s_{nc} V = -\frac{i}{2} L_V (\bar{\Lambda} + \Lambda) + \frac{i}{2} L_V \coth_{\star} \left(\frac{L_V}{2} \right) (\bar{\Lambda} - \Lambda), \quad L_V = [V, \quad]_{\star}, \quad s_{nc} \Lambda = i\Lambda \star \Lambda,$$

where Λ now denotes an infinitesimal Grassmann chiral superfield. Let v and λ denote, respectively, an ordinary U(1) vector and an ordinary U(1) ghost superfields. In keeping with the ideas underlying the Seiberg-Witten map, to obtain an ordinary theory dual of noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills, one should first express the U(1) noncommutative superfields V and Λ as functions of v and λ , and their susy covariant derivatives, in such a way that ordinary BRS orbits are mapped into noncommutative BRS orbits. This is achieved by solving the Seiberg-Witten-map equations for U(1) superfields. These

equations read

$$\begin{aligned} s_{nc} \Lambda[\lambda, v] &= s\Lambda[\lambda, v], & \Lambda, \lambda & \text{chiral,} \\ s_{nc} V[v] &= sV[v], & V, v & \text{real.} \end{aligned}$$

The symbol s denotes the ordinary U(1) BRS operator, which acts on the ordinary superfields as follows

$$sv = i(\bar{\lambda} - \lambda), \quad s\lambda = 0. \quad (2.2)$$

Expanding the noncommutative fields in powers of $h\omega^{mn}$,

$$\Lambda = \lambda + h\Lambda^{(1)} + O(h^2), \quad V = v + hV^{(1)} + O(h^2), \quad (2.3)$$

one gets the following equations for the first order contributions:

$$\begin{aligned} s\Lambda^{(1)} &= \frac{1}{32} \omega_{\alpha\beta} \partial^{\dot{\kappa}\alpha} \lambda \partial_{\dot{\kappa}}^{\beta} \lambda + \frac{1}{32} \omega_{\dot{\alpha}\dot{\beta}} \partial^{\dot{\alpha}\kappa} \lambda \partial_{\dot{\beta}}^{\kappa} \lambda, \\ sV^{(1)} &= -\frac{1}{32} \omega^{\alpha\beta} \partial^{\dot{\kappa}}_{\alpha} v \lambda \partial_{\dot{\kappa}\beta} (\lambda + \bar{\lambda}) - \frac{1}{32} \omega^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\alpha}}^{\kappa} v \partial_{\dot{\beta}\kappa} (\lambda + \bar{\lambda}) + i(\bar{\Lambda}^{(1)} - \Lambda^{(1)}), \end{aligned} \quad (2.4)$$

where we used — see appendix A for notation — the following relations between vector indices (Latin letters) and spinor indices (Greek letters):

$$\begin{aligned} \partial_{\dot{\alpha}\beta} &= (\bar{\sigma}^m)_{\dot{\alpha}\beta} \partial_m, \\ \omega^{mn} &= -\frac{1}{16} (\sigma^{mn})^{\alpha\beta} \omega_{\alpha\beta} + \frac{1}{16} (\bar{\sigma}^{mn})^{\dot{\alpha}\dot{\beta}} \omega_{\dot{\alpha}\dot{\beta}}, \\ \omega^{\rho\sigma} &= -2(\sigma^{mn})^{\rho\sigma} \omega_{mn}, & \omega^{\dot{\rho}\dot{\sigma}} &= 2(\bar{\sigma}^{mn})^{\dot{\rho}\dot{\sigma}} \omega_{mn}. \end{aligned}$$

One should first look for solutions to eq. (2.4) that would allow us to make contact with the Seiberg-Witten map — called the standard Seiberg-Witten map — as introduced in ref. [1]. In looking for these solutions the first obstacle one stumbles on is the fact that, at first order in $h\omega^{mn}$, the standard Seiberg-Witten map is never the $\theta^\alpha \bar{\theta}^{\dot{\alpha}}$ component of a real superfield, with no free spinor indices, which is a polynomial in v and its susy covariant derivatives $D_\alpha, \bar{D}_{\dot{\alpha}}, \partial_{\alpha\dot{\beta}}$. This fact, that has not been properly discussed in the literature as yet, contradicts the claim made in ref. [14] that the standard Seiberg-Witten map can be supersymmetrised at first order in $h\omega^{mn}$, i.e., that at first order in $h\omega^{mn}$ a dimensionless real polynomial in v and its susy derivatives with no free spinor indices can be constructed so that its $\theta^\alpha \bar{\theta}^{\dot{\alpha}}$ component is the standard Seiberg-Witten map.

The BRS transformations with nonstandard normalisations inherited by the gauge fields A_m — noncommutative — and a_m — ordinary — from the superfield gauge transformations in eqs. (2.1) and (2.2) read $s_{nc} A_m = -2\partial_m Z - i[A_m, Z]_*$, $s a_m = -2\partial_m z$. For these BRS transformations the first-order-in- $h\omega^{mn}$ standard Seiberg-Witten map of ref. [1] runs thus

$$A_l^{(1)\text{st}} = \frac{1}{2} \omega^{mn} \left(a_m \partial_n a_l - \frac{1}{2} a_m \partial_l a_n \right). \quad (2.5)$$

Let us now show that this $A_l^{(1)\text{st}}$ is not the $\theta^\beta (\bar{\sigma}_l)_{\dot{\alpha}\beta} \bar{\theta}^{\dot{\alpha}}$ component of a dimensionless real polynomial with no free spinor indices made out of v and its susy derivatives. Since $A_l^{(1)\text{st}}$ is

quadratic in a_m , it suffices to consider the most general, \tilde{V} , dimensionless real polynomial in v and its susy derivatives with no free spinor indices which is linear in ω^{mn} and quadratic in v . \tilde{V} is given by

$$\tilde{V} = \sum_{i=1}^5 (x_i \text{Re } t_i + y_i \text{Im } t_i), \quad x_i, y_i \in \mathbb{R},$$

where $\text{Re } t_i$ and $\text{Im } t_i$ denote, respectively, the real and imaginary parts of t_i , and x_i and y_i are arbitrary real coefficients. $\{t_i\}_{i=1\dots 5}$ denotes the following set of monomials

$$\begin{aligned} t_1 &= \omega^{\alpha\beta} \partial_{\alpha\dot{\beta}} \bar{D}^{\dot{\beta}} D_{\beta} v v, \quad t_2 = \omega^{\alpha\beta} \partial_{\alpha\dot{\beta}} \bar{D}^{\dot{\beta}} v D_{\beta} v, \quad t_3 = i \omega^{\alpha\beta} \bar{D}^2 D_{\alpha} v D_{\beta} v, \quad t_4 = \omega^{\alpha\beta} \partial_{\alpha\dot{\beta}} D_{\beta} v \bar{D}_{\dot{\beta}} v, \\ t_5 &= \omega^{\alpha\beta} D_{\alpha} \bar{D}^{\dot{\alpha}} v \partial_{\beta\dot{\alpha}} v. \end{aligned} \quad (2.6)$$

For the reader's sake we also display the complex conjugates, \bar{t}_i , $i = 1 \dots 5$, of the previous monomials:

$$\begin{aligned} \bar{t}_1 &= -\omega^{\dot{\alpha}\dot{\beta}} \partial_{\beta\dot{\alpha}} D^{\beta} \bar{D}_{\dot{\beta}} v v, \quad \bar{t}_2 = -\omega^{\dot{\alpha}\dot{\beta}} \partial_{\beta\dot{\alpha}} D^{\beta} v \bar{D}_{\dot{\beta}} v, \quad \bar{t}_3 = -i \omega^{\dot{\alpha}\dot{\beta}} D^2 \bar{D}_{\dot{\alpha}} v \bar{D}_{\dot{\beta}} v, \quad \bar{t}_4 = -\omega^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\alpha}} \bar{D}_{\dot{\beta}} v D_{\beta} v, \\ \bar{t}_5 &= -\omega^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\alpha}} D^{\alpha} v \partial_{\alpha\dot{\beta}} v. \end{aligned}$$

Let us now show that for no choice of x_i and y_i the following equation will hold

$$\frac{1}{4} (\sigma_m)^{\beta\dot{\alpha}} [\bar{D}_{\dot{\alpha}}, D_{\beta}] \tilde{V} |_{\theta=\bar{\theta}=0, aa} = A_m^{(1)\text{st}}. \quad (2.7)$$

By aa , we mean that only the contributions quadratic in a_m are kept. Now, it can be seen that the a_m -dependent part of the terms $\text{Im } t_i$ always involve contractions with the Levi-Civita symbol $\epsilon^{mnr s}$, which never occur in eq. (2.5) — recall that ω^{mn} is real and that our noncommutative space-time has got Minkowski signature. Hence, the y_i will be of no avail to make eq. (2.5) hold and thus we shall only worry about the contributions coming from $\text{Re } t_i$. Introducing the notation $\frac{1}{4} (\sigma_m)^{\beta\dot{\alpha}} [\bar{D}_{\dot{\alpha}}, D_{\beta}] \text{Re } t_i \equiv \tilde{A}_m^{[i]}$ and after some computations one finds that

$$\begin{aligned} \tilde{A}_l^{[1]} &= -4 \omega^{mn} f_{mn} a_l, \\ \tilde{A}_l^{[2]} &= -2 \omega^{mn} (f_{mn} a_l + 2 \eta_{ml} a_n (\partial a) + 2 f_{ml} a_n - 2 \eta_{ln} f_{mk} a^k), \\ \tilde{A}_l^{[3]} &= -8 \omega^{mn} (f_{mn} a_l - 2 f_{lm} a_n - 2 \eta_{lm} f_{kn} a^k), \\ \tilde{A}_l^{[4]} &= -2 \omega^{mn} (f_{mn} a_l - 2 f_{ln} a_m + 4 \partial_m a_l a_n - 2 \eta_{ln} f_{mk} a^k + 4 \eta_{lm} \partial^k a_n a_k + 2 \eta_{ln} (\partial a) a_m), \\ \tilde{A}_l^{[5]} &= 16 \omega^{mn} a_m \partial_n a_l. \end{aligned}$$

Finally, eq. (2.7) boils down to

$$\sum x_i \tilde{A}_l^{[i]} = \frac{1}{2} \omega^{mn} \left(a_m \partial_n a_l - \frac{1}{2} a_m \partial_l a_n \right),$$

which has no solution since, in spite of the fact that the terms that occur on its r.h.s. can be obtained by choosing several values of the x_i , there always appear undesired extra

terms involving contractions of the type $\omega^{mn}\eta_{lm}$. Notice that the ambiguity [37] of the Seiberg-Witten map cannot be taken advantage of to fix this situation, for this ambiguity, in the U(1) case, is linear in a_m .

In searching for solutions to eq. (2.4), the second difficulty one meets is that, as shown in ref. [19], $\Lambda^{(1)}$ cannot be a polynomial in v , λ and its susy derivatives, since $\Lambda^{(1)}$ is chiral. Thus one is led to look for nonlocal solutions to eq. (2.4), i.e., solutions that are not polynomials in the ordinary superfields and their susy derivatives. To avoid the inconsistencies that usually arise in theories with gauge independent nonlocal terms, one may look for solutions to eq. (2.4) whose nonlocal contributions vanish in a given gauge. Since both the chiral and antichiral projections of v , namely, $v_+ \equiv P_+v$ and $v_- \equiv P_-v$, with $P_+ = \frac{1}{16\pi^2} \frac{\bar{D}^2 D^2}{\square}$ and $P_- = \frac{1}{16\pi^2} \frac{D^2 \bar{D}^2}{\square}$, vanish in the susy Landau gauge $D^2 \bar{D}^2 v = \bar{D}^2 D^2 v = 0$, and since projecting v into its chiral part may help find a chiral $\Lambda^{(1)}$, it is natural — and the next simplest ansatz to that of local solutions — to look for solutions to eq. (2.4) that are polynomials in λ, v, v_+, v_- and their susy covariant derivatives. We shall further assume that $\Lambda^{(1)}$ is linear in v, v_{\pm} , and that $V^{(1)}$ is at most quadratic in v, v_{\pm} ; the rationale for these assumptions is that the corresponding first-order-in- $\hbar\omega^{mn}$ contributions to the standard Seiberg-Witten maps are, respectively, linear and quadratic in a_m . Let us introduce some more notation: $\bar{v} \equiv v - v_+ - v_-$ — of course, $s\bar{v} = 0$. A lengthy computation yields the following family of solutions to eq. (2.4):

$$\Lambda^{(1)} = \frac{i}{32} \omega^{\alpha\beta} \partial^{\dot{\alpha}}{}_{\alpha} v_+ \partial_{\dot{\alpha}\beta} \lambda + \frac{i}{32} \omega^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\alpha}}{}^{\alpha} v_+ \partial_{\alpha\dot{\beta}} \lambda + x \omega^{\alpha\beta} \bar{D}^2 (D_{\alpha} \bar{v} D_{\beta} \lambda), \quad (2.8)$$

$$\begin{aligned} V^{(1)} = & x \omega^{\alpha\beta} \bar{D}^2 (D_{\alpha} \bar{v} D_{\beta} v) + \bar{x} \omega^{\dot{\alpha}\dot{\beta}} D^2 (\bar{D}_{\dot{\alpha}} \bar{v} \bar{D}_{\dot{\beta}} v) \\ & + \frac{i}{32} \omega^{\alpha\beta} [\partial^{\dot{\alpha}}{}_{\alpha} (v - v_-) \partial_{\dot{\alpha}\beta} (v - v_+)] \\ & - \frac{i}{32} \omega^{\dot{\alpha}\dot{\beta}} [\partial_{\dot{\alpha}}{}^{\alpha} (v - v_+) \partial_{\dot{\beta}\alpha} (v - v_-)] + \mathcal{X}, \quad s\mathcal{X} = 0. \end{aligned} \quad (2.9)$$

x is an arbitrary constant parametrising the ambiguity in the map for Λ ; it must be imaginary if one wants to avoid — as happens in standard Seiberg-Witten map case — parity violating terms — contributions involving contractions with the $\epsilon^{mnr s}$ symbol — in the map for the component field a_m that otherwise will make the noncommutative and ordinary gauge fields behave not in the same way under parity. \mathcal{X} represents the ambiguity in the map for the real superfield V ; it is given by the most general linear combination of terms constructed from \bar{v} and susy covariant derivatives $D_{\alpha}, \bar{D}_{\dot{\alpha}}, \partial_{\dot{\alpha}\beta}$, i.e., a linear combination of the real and imaginary parts of the terms appearing in eq. (2.6), with v substituted by \bar{v} . \mathcal{X} can be interpreted as a field redefinition of v . Our solutions for $\Lambda^{(1)}$ and $V^{(1)}$ include the particular solution found in ref. [15].

In the case of the map for V , the x -dependent terms can be gauged away by performing a gauge transformation of v , since they can be written as the difference of a chiral and an antichiral term. It is plain that in the supersymmetric Landau gauge the Seiberg-Witten map above is local and $V^{(1)}$ is given by the most general local expression quadratic in v that one can write; this is a very welcomed feature of the map in regards with renormalisability issues [38–40].

In refs. [17, 18] the standard Seiberg-Witten map was used to construct an ordinary — i.e., on ordinary Minkowski space-time — field theory that is dual to U(1) noncommutative SYM theory formulated in the Wess-Zumino gauge. This ordinary dual theory is formulated in terms of the “susy” gauge multiplet (a_m, λ_α, d) , which undergoes ordinary U(1) transformations but whose “susy” transformations are a sum of the ordinary susy transformations plus nonlinear ω^{mn} -dependent terms — this is why for the time being we write “susy” and not susy; we shall show that these comas can be removed in section 3. Since it is one of the purposes of this paper to relate the ordinary dual theory obtained from noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills by using the Seiberg-Witten map for superfields — see eqs. (2.3), (2.8) and (2.9) — with the dual ordinary theory obtained from the latter noncommutative theory as in refs. [17, 18], we shall need to gauge transform to the Wess-Zumino gauge the noncommutative scalar superfield $V[v]$ defined in eqs. (2.3) and (2.9). Let us stress first that if v^{WZ} denotes a general ordinary real scalar superfield in the Wess-Zumino gauge, then its noncommutative image, $V[v^{\text{WZ}}]$, given by the Seiberg-Witten map in eqs. (2.3) and (2.9), is not a noncommutative real scalar superfield in the Wess-Zumino gauge. But, of course, one can further gauge transform this $V[v^{\text{WZ}}]$ to a new noncommutative scalar superfield $V^{\text{WZ}}[a_m, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, d]$ which is in the Wess-Zumino gauge — $a_m, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}$ and d are the components of v^{WZ} . Indeed,

$$e^{V^{\text{WZ}}[a_m, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, d]} = e_{\star}^{i\bar{\Lambda}^{\text{WZ}}} \star e_{\star}^{V[v^{\text{WZ}}]} \star e_{\star}^{-i\Lambda^{\text{WZ}}}, \quad (2.10)$$

for a Λ^{WZ} which is linear in $h\omega^{mn}$, leads to

$$\begin{aligned} V^{\text{WZ}}[a_m, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, d] &= v^{\text{WZ}} + hV^{(1)}[v^{\text{WZ}}] + ih(\bar{\Lambda}^{\text{WZ}} - \Lambda^{\text{WZ}}) + O(h^2), \\ \Lambda^{\text{WZ}} &= -\frac{i}{2}C^{(1)}(y) - i\theta^\alpha\Psi_\alpha^{(1)}(y) - \frac{i}{2}\theta^2 F^{(1)}(y), \quad y^m = x^m - i\theta^\alpha\bar{\sigma}_{\dot{\beta}\alpha}^m\bar{\theta}^{\dot{\beta}}. \end{aligned} \quad (2.11)$$

$C^{(1)}(x)$, $\Psi_\alpha^{(1)}(x)$ and $F^{(1)}(x)$ are the lowest components of $V^{(1)}[v^{\text{WZ}}]$, the latter defined by eq. (2.9):

$$\begin{aligned} V^{(1)}[v^{\text{WZ}}] &= C^{(1)} + \theta^\alpha\Psi_\alpha^{(1)} + \bar{\theta}^{\dot{\alpha}}\bar{\Psi}_{\dot{\alpha}}^{(1)} + \frac{1}{2}\theta^2 F^{(1)} + \frac{1}{2}\bar{\theta}^2\bar{F}^{(1)} + \theta^\alpha\bar{\theta}^{\dot{\beta}}A_{\dot{\beta}\alpha}^{(1)} + \frac{1}{2}\theta^2\bar{\theta}^{\dot{\alpha}}\bar{\Lambda}'_{\dot{\alpha}}{}^{(1)} + \frac{1}{2}\bar{\theta}^2\theta^\alpha\Lambda'_\alpha{}^{(1)} \\ &\quad + \frac{1}{4}\bar{\theta}^2\theta^2 D'^{(1)}, \\ \Lambda'_\alpha{}^{(1)} &= \Lambda_\alpha^{(1)} - i\bar{\sigma}_{\dot{\beta}\alpha}^m\partial_m\bar{\Psi}^{(1)\dot{\beta}}, \quad D'^{(1)} = D^{(1)} + \square C^{(1)}, \\ v^{\text{WZ}} &= \theta^\alpha\bar{\theta}^{\dot{\beta}}a_{\dot{\beta}\alpha} + \frac{1}{2}\theta^2\bar{\theta}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}} + \frac{1}{2}\bar{\theta}^2\theta^\alpha\lambda_\alpha + \frac{1}{4}\bar{\theta}^2\theta^2 d. \end{aligned} \quad (2.12)$$

For $x = 0$ and $\mathcal{X} = 0$, the components of $V^{(1)}[v^{\text{WZ}}]$ read

$$\begin{aligned} C^{(1)} &= -\frac{\omega^{\alpha\beta}}{256}\frac{\partial_{\dot{\alpha}}^\alpha}{\square}d\frac{\partial_{\dot{\beta}\alpha}}{\square}\partial a + c.c., \\ \Psi_\sigma^{(1)} &= \frac{\omega^{\alpha\beta}}{256}\frac{\partial_{\dot{\alpha}}^\alpha}{\square}(d - 2i\partial a)\frac{\partial_{\dot{\beta}\alpha}\partial_{\dot{\rho}\sigma}}{\square}\bar{\lambda}^{\dot{\rho}} + \frac{\omega^{\dot{\alpha}\dot{\beta}}}{256}\frac{\partial_{\dot{\alpha}}^\alpha}{\square}(d - 2i\partial a)\frac{\partial_{\dot{\beta}\alpha}\partial_{\dot{\rho}\sigma}}{\square}\bar{\lambda}^{\dot{\rho}}, \\ F^{(1)} &= 0, \end{aligned}$$

$$A_{\dot{\beta}\dot{\gamma}}^{(1)} = \frac{\omega^{\alpha\beta}}{256} \left[8\partial_{\dot{\alpha}}^{\dot{\alpha}} a_{\dot{\beta}\dot{\gamma}} \frac{\partial_{\dot{\alpha}\beta}}{\square} \partial a + 4\frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}}{\square} \partial a \frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\beta}\dot{\gamma}}}{\square} \partial a + \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}}{\square} d \frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\beta}\dot{\gamma}}}{\square} d - 2i\frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}\partial_{\dot{\beta}\sigma}}{\square} \lambda^{\sigma} \frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\gamma}\dot{\sigma}}}{\square} \bar{\lambda}^{\dot{\sigma}} \right] \quad (2.13)$$

+ (c.c.) $_{(\beta\leftrightarrow\gamma)}$,

$$\Lambda_{\rho}^{(1)} = \frac{1}{128} \left[-4\omega^{\alpha\beta} \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}}{\square} \partial a \partial_{\dot{\alpha}\beta} \lambda_{\rho} - 4\omega^{\dot{\alpha}\dot{\beta}} \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}}{\square} \partial a \partial_{\dot{\beta}\alpha} \lambda_{\rho} + 2\omega^{\alpha\beta} \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}\partial_{\dot{\sigma}}^{\dot{\rho}}}{\square} \lambda^{\sigma} \partial_{\dot{\alpha}\beta} \left(a_{\dot{\rho}\rho} - \frac{\partial_{\dot{\rho}\rho}}{\square} \partial a \right) + \right. \\ \left. + 2\omega^{\dot{\alpha}\dot{\beta}} \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}\partial_{\dot{\sigma}}^{\dot{\rho}}}{\square} \lambda^{\sigma} \partial_{\dot{\beta}\alpha} \left(a_{\dot{\rho}\rho} - \frac{\partial_{\dot{\rho}\rho}}{\square} \partial a \right) + i\omega^{\alpha\beta} \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}\partial_{\dot{\sigma}}^{\dot{\rho}}}{\square} \lambda^{\sigma} \frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\rho}\rho}}{\square} d + i\omega^{\dot{\alpha}\dot{\beta}} \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}\partial_{\dot{\sigma}}^{\dot{\rho}}}{\square} d \frac{\partial_{\dot{\beta}\alpha}\partial_{\dot{\rho}\sigma}}{\square} \lambda^{\sigma} \right],$$

$$D^{(1)} = \frac{\omega^{\alpha\beta}}{128} \left[4\partial_{\dot{\alpha}}^{\dot{\alpha}} d \frac{\partial_{\dot{\alpha}\beta}}{\square} \partial a + \frac{\partial_{\dot{\alpha}}^{\dot{\alpha}}\partial_{\dot{\sigma}}^{\dot{\rho}}}{\square} \lambda^{\sigma} \partial_{\dot{\alpha}\beta} \bar{\lambda}_{\dot{\rho}} + \partial_{\dot{\alpha}}^{\dot{\alpha}} \lambda^{\sigma} \frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\sigma}\sigma}}{\square} \bar{\lambda}^{\dot{\sigma}} + 2\partial_{\dot{\alpha}}^{\dot{\alpha}} \left(a^{\sigma\rho} - \frac{\partial^{\sigma\rho}}{\square} \partial a \right) \frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\rho}\sigma}}{\square} d \right] + c.c.$$

In the previous equations, (c.c.) denotes complex conjugate and (c.c.) $_{|\beta\leftrightarrow\gamma}$ denotes complex conjugate with indices β and γ exchanged (hermitian conjugation); for example $\bar{\sigma}_{\dot{\beta}\dot{\gamma}} + (c.c.)_{|\beta\leftrightarrow\gamma} = 2\bar{\sigma}_{\dot{\beta}\dot{\gamma}}$.

Taking into account eqs. (2.11), (2.12) and (2.13), one concludes that

$$V^{\text{WZ}}[a_m, \lambda_{\alpha}, \bar{\lambda}_{\dot{\alpha}}, d] = \theta^{\alpha}\bar{\theta}^{\dot{\beta}} A_{\dot{\beta}\alpha} + \frac{1}{2}\theta^2\bar{\theta}^{\dot{\alpha}} \bar{\Lambda}_{\dot{\alpha}} + \frac{1}{2}\bar{\theta}^2\theta^{\alpha} \Lambda_{\alpha} + \frac{1}{4}\bar{\theta}^2\theta^2 D, \quad (2.14)$$

$$A_{\dot{\beta}\alpha} = a_{\dot{\beta}\alpha} + hA_{\dot{\beta}\alpha}^{(1)} + O(h^2), \quad \Lambda_{\alpha} = \lambda_{\alpha} + h\Lambda_{\alpha}^{(1)} + O(h^2), \quad D = d + hD^{(1)} + O(h^2),$$

where $A_{\dot{\beta}\alpha}^{(1)}$, $\Lambda_{\alpha}^{(1)}$ and $D^{(1)}$ are the same as for $V[v^{\text{WZ}}]$ and thus given in eq. (2.13). Let us stress that $V[v^{\text{WZ}}]$ and $V^{\text{WZ}}[a_m, \lambda_{\alpha}, \bar{\lambda}_{\dot{\alpha}}, d]$ define the same theory since they are related by a noncommutative gauge transformation.

We shall close this section by recalling that the ambiguity \mathcal{X} in the Seiberg-Witten map in eq. (2.9) has no physical consequences since it is a local field redefinition of the ordinary vector superfield, hence we shall set it to zero from now on.

3. Ordinary duals of noncommutative $U(N)$ $\mathcal{N} = 1$ superYang-Mills theory under the standard Seiberg-Witten map

In refs. [17] and [18], the standard Seiberg-Witten map was used to map noncommutative $U(1)$ SYM theory in the Wess-Zumino gauge to an ordinary gauge theory with $U(1)$ symmetry. This construction can be generalised to noncommutative $U(N)$ gauge groups as we shall do next. The construction we are about to develop may be of relevance in studying some of the physical implications of the models proposed in refs. [41–45] and [46].

Our supersymmetric noncommutative field theory will have the following field content: a noncommutative gauge $\mathcal{N} = 1$ supermultiplet, $(A_m, \Lambda_{\alpha}, D)$. The fields A_m, Λ_{α}, D are valued in the Lie algebra of $U(N)$ in the fundamental representation. If $Z(x) = Z^a(x) T^a$ denotes an infinitesimal function valued in the Lie algebra of $U(N)$ in the fundamental representation, with $Z^a(x)$ being ghost fields, our theory will be invariant under the following noncommutative BRS transformations:

$$s_Z^{nc} A_m = -\hat{D}_m Z = -(\partial_m Z + i[A_m, Z]_{\star}), \quad s_Z^{nc} \Lambda_{\alpha} = -i[\Lambda_{\alpha}, Z]_{\star}, \quad s_Z^{nc} D = -i[D, Z]_{\star}.$$

In addition to the BRS symmetry just defined, our $U(N)$ noncommutative gauge theory will be invariant under the following supersymmetry transformations:

$$\widehat{\delta}_\epsilon A_m = \frac{1}{4}\epsilon\sigma_m\bar{\Lambda} - \frac{1}{4}\bar{\epsilon}\bar{\sigma}_m\Lambda, \quad \widehat{\delta}_\epsilon\Lambda_\alpha = -\epsilon_\alpha D + 2i\epsilon_\gamma(\sigma^{mn})^\gamma{}_\alpha F_{mn}, \quad \widehat{\delta}_\epsilon D = i\bar{\epsilon}\bar{\sigma}^m\hat{D}_m\Lambda + i\epsilon\sigma^m\hat{D}_m\bar{\Lambda}, \quad (3.1)$$

where $F_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n]_\star$ and $\hat{D}_m = \partial_m + i[A_m, \quad]_\star$. These supersymmetry transformations are linear modulo noncommutative gauge transformations, hence the noncommutative multiplets of our theory carry a linear representation of the supersymmetry algebra: of course, there is a formulation of our theory in terms of superfields, each multiplet above constituting the components of the appropriate superfield in the Wess-Zumino gauge.

Let \tilde{a}_m , $\tilde{\lambda}_\alpha$ and \tilde{d} stand, respectively, for the ordinary counterparts, under the standard Seiberg-Witten map, of the noncommutative fields A_m , Λ_α and D introduced above. Then, up to first order in $\hbar\omega^{mn}$, the standard Seiberg-Witten map for our theory is given by the following equations

$$\begin{aligned} A_m[\tilde{a}_n] &= \tilde{a}_m + \frac{\hbar}{4}\omega^{nl}\{\tilde{a}_n, \partial_l\tilde{a}_m + \tilde{f}_{lm}\} + O(\hbar^2), \\ \Lambda_\alpha[\tilde{a}_m, \tilde{\lambda}_\alpha] &= \tilde{\lambda}_\alpha + \frac{\hbar}{4}\omega^{mn}\{\tilde{a}_m, 2D_n\tilde{\lambda}_\alpha - i[\tilde{a}_n, \tilde{\lambda}_\alpha]\} + O(\hbar^2), \\ D[\tilde{a}_m, \tilde{d}] &= \tilde{d} + \frac{\hbar}{4}\omega^{mn}\{\tilde{a}_m, 2D_n\tilde{d} - i[\tilde{a}_n, \tilde{d}]\} + O(\hbar^2), \end{aligned} \quad (3.2)$$

where $\tilde{f}_{nl} = \partial_n\tilde{a}_l - \partial_l\tilde{a}_n + i[\tilde{a}_n, \tilde{a}_l]$, $D_m = \partial_m + i[\tilde{a}_m, \quad]$. By construction the Seiberg-Witten map defined in eq. (3.2) maps infinitesimal gauge orbits of the ordinary theory into infinitesimal gauge orbits of the noncommutative theory. Indeed, if the noncommutative field $\mathcal{U}[\tilde{a}_m, u]$ is the image under the Seiberg-Witten map of u , then

$$\mathcal{U}[\tilde{a}_m, u] + \kappa s_{nc}\mathcal{U}[\tilde{a}_m, u] = \mathcal{U}[\tilde{a}_m + \kappa\tilde{s}\tilde{a}_m, u + \kappa\tilde{s}u], \quad (3.3)$$

κ being the infinitesimal BRS Grassmann parameter and \tilde{s} being the ordinary BRS operator which acts on our fields with tilde as follows:

$$\tilde{s}_z\tilde{a}_m = -D_m z = -(\partial_m z + i[\tilde{a}_m, z]), \quad \tilde{s}_z\Lambda_\alpha = -i[\Lambda_\alpha, z], \quad \tilde{s}_z\tilde{d} = -i[\tilde{d}, z].$$

Of course, in eq. (3.3), Z in s_{nc} and z in \tilde{s} are not independent, but related by

$$Z = z + \frac{\hbar}{4}\omega^{mn}\{\tilde{a}_m, \partial_n z\}. \quad (3.4)$$

We have seen that the Seiberg-Witten map in eq. (3.2) maps a theory on ordinary space-time having an ordinary $U(N)$ gauge symmetry to a noncommutative $U(N)$ gauge theory having, therefore, a noncommutative gauge symmetry. But, this noncommutative gauge theory is further a supersymmetric theory and its fields carry a linear — the supersymmetric transformations in eq. (3.1) are linear modulo noncommutative gauge transformations — representation of the supersymmetry algebra, i.e., the commutator of two supersymmetry transformations acting on a noncommutative field, \mathcal{U} , closes on space-time translations modulo a noncommutative gauge transformation:

$$[\widehat{\delta}_\xi, \widehat{\delta}_\eta]\mathcal{U}(x) = -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\partial_m\mathcal{U}(x) + \delta_\Omega^{(ncgauge)}\mathcal{U}(x) \equiv P\mathcal{U}(x) + \delta_\Omega^{(ncgauge)}\mathcal{U}(x). \quad (3.5)$$

$\mathcal{U}(x)$ denotes any of the noncommutative fields of our noncommutative theory. $\delta_\Omega^{(ncgauge)} \mathcal{U}(x)$ is a noncommutative gauge transformations with $\Omega(x) = -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta}) A_m(x)$. The next issue to be addressed is whether there exist transformations of the ordinary fields that occur in the Seiberg-Witten map in eq. (3.2) that give rise to the supersymmetry transformations of the corresponding noncommutative fields that we have just discussed. The answer to this problem is that there exist such transformations since we are dealing with $U(N)$ in the fundamental and antifundamental representations. Indeed, we shall look for infinitesimal variations, $\tilde{\delta}_\epsilon u$, of the ordinary fields in eq. (3.2), collectively denoted by u , such that

$$\mathcal{U}[\tilde{a}_m, u] + \widehat{\delta}_\epsilon \mathcal{U}[\tilde{a}_m, u] = \mathcal{U}[\tilde{a}_m + \tilde{\delta}_\epsilon \tilde{a}_m, u + \tilde{\delta}_\epsilon u], \quad (3.6)$$

where $\widehat{\delta}_\epsilon \mathcal{U}[\tilde{a}_m, u]$ is defined in eq. (3.1). Since we understand the Seiberg-Witten map as a formal power series expansion in $h\omega^{mn}$, it turns out that $\tilde{\delta}_\epsilon u$ can be obtained from eq. (3.6) as a formal power series expansion in $h\omega^{mn}$, provided that the representation of the gauge group that one considers satisfies: $L_1 \cdot L_2$ belongs to its Lie algebra in the corresponding representation, if L_1 and L_2 do. As pointed out in ref. [35], this condition restricts the type of gauge group to $U(N)$ groups, or products of them, and the type of irreducible representation to the fundamental, antifundamental, adjoint and bi-fundamental. Up to first order in $h\omega^{mn}$, we have

$$\begin{aligned} \tilde{\delta}_\epsilon \tilde{a}_m &= \frac{1}{4}\epsilon\sigma_m\bar{\lambda} - \frac{1}{4}\bar{\epsilon}\bar{\sigma}_m\tilde{\lambda} + \frac{h}{16}\omega^{nl} \left[\{\tilde{a}_n, 2D_l(\epsilon\sigma_m\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_m\tilde{\lambda}) - i[\tilde{a}_l, \epsilon\sigma_m\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_m\tilde{\lambda}] \right. \\ &\quad \left. - \{\epsilon\sigma_n\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_n\tilde{\lambda}, \partial_l\tilde{a}_m + \tilde{f}_{lm}\} - \{\tilde{a}_n, \partial_l(\epsilon\sigma_m\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_m\tilde{\lambda}) \right. \\ &\quad \left. + D_l(\epsilon\sigma_m\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_m\tilde{\lambda}) - D_m(\epsilon\sigma_l\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_l\tilde{\lambda}) \right], \\ \tilde{\delta}_\epsilon \tilde{\lambda}_\alpha &= -\epsilon_\alpha \tilde{d} + 2i\epsilon_\gamma(\sigma^{mn})^\gamma{}_\alpha \tilde{f}_{mn} + \frac{h}{4}\omega^{nl} \left[-\frac{1}{4}\{\epsilon\sigma_n\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_n\tilde{\lambda}, 2D_l\tilde{\lambda}_\alpha - i[\tilde{a}_l, \tilde{\lambda}_\alpha]\} \right. \\ &\quad \left. - i\epsilon_\gamma(\sigma^{mk})^\gamma{}_\alpha (4\{\tilde{f}_{mn}, \tilde{f}_{kl}\} - 2\{\tilde{a}_n, D_l\tilde{f}_{mk} + \partial_l\tilde{f}_{mk}\}) \right. \\ &\quad \left. - \{\tilde{a}_n, 4iD_l(\epsilon_\gamma(\sigma^{mk})^\gamma{}_\alpha \tilde{f}_{mk}) + 2[\tilde{a}_l, \epsilon_\gamma(\sigma^{mk})^\gamma{}_\alpha \tilde{f}_{mk}] + \frac{i}{4}[\epsilon\sigma_l\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_l\tilde{\lambda}, \tilde{\lambda}_\alpha]\} \right], \\ \tilde{\delta}_\epsilon \tilde{d} &= i\bar{\epsilon}\bar{\sigma}^m D_m \tilde{\lambda} + i\epsilon\sigma^m D_m \bar{\lambda} + \frac{h}{4}\omega^{nl} \left[2i\{\tilde{f}_{mn}, \bar{\epsilon}\bar{\sigma}^m D_l \tilde{\lambda} + \epsilon\sigma^m D_l \bar{\lambda}\} \right. \\ &\quad \left. + i\{\tilde{a}_n, (\partial_l + D_l)(\bar{\epsilon}\bar{\sigma}^m D_m \tilde{\lambda} + \epsilon\sigma^m D_m \bar{\lambda})\} - \frac{1}{4}\{\epsilon\sigma_n\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_n\tilde{\lambda}, 2D_l\tilde{d} - i[\tilde{a}_l, \tilde{d}]\} \right. \\ &\quad \left. - \{\tilde{a}_n, 2D_l(i\bar{\epsilon}\bar{\sigma}^m D_m \tilde{\lambda} + i\epsilon\sigma^m D_m \bar{\lambda}) \right. \\ &\quad \left. - i[\tilde{a}_l, i\bar{\epsilon}\bar{\sigma}^m D_m \tilde{\lambda} + i\epsilon\sigma^m D_m \bar{\lambda}] + \frac{i}{4}[\epsilon\sigma_l\bar{\lambda} - \bar{\epsilon}\bar{\sigma}_l\tilde{\lambda}, \tilde{d}] \right]. \end{aligned} \quad (3.7)$$

We have thus worked out, up to first order in $h\omega^{mn}$, the infinitesimal variations of the ordinary fields that give rise through the Seiberg-Witten map in eq. (3.2) to the linearly realised supersymmetric transformations — see eq. (3.1) — of the noncommutative fields. Of course, if we set $h = 0$, these infinitesimal variations of the ordinary fields boil down to the ordinary supersymmetry transformations of an ordinary gauge theory in the Wess-Zumino gauge. However, the contributions of order $h\omega^{mn}$ are nonlinear modulo gauge transformations, and tell us that unlike for gauge symmetries the standard Seiberg-Witten

map in eq. (3.2) does not transmute supersymmetry transformations of the ordinary fields realising supersymmetry linearly into supersymmetry transformations of the noncommutative fields also realising supersymmetry linearly. The question then arises as to whether the nonlinear transformations in eq. (3.7) realise a — nonlinear — representation of supersymmetry in the sense that the commutator of two such transformations on ordinary fields closes on space-time translations modulo ordinary gauge transformations. If we can answer the question in the affirmative — which we shall, at any order in $h\omega^{mn}$ —, we will be entitled to call the transformations in eq. (3.7) supersymmetry transformations. This issue has never been discussed in the literature, although the U(1) version of the transformations in eq. (3.7) have been called supersymmetry transformations. Let us show that if $\tilde{\delta}_\epsilon u$ is an infinitesimal transformation satisfying eq. (3.6), then

$$[\tilde{\delta}_\xi, \tilde{\delta}_\eta]u(x) = -2i(\eta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\eta})\partial_m u(x) + \delta_{g(x)}^{(gauge)} u(x) \equiv (P + \delta_{g(x)}^{(gauge)}) u(x), \quad (3.8)$$

where $g(x)$ is the inverse image of $\Omega(x)$ in eq. (3.5) under the Seiberg-Witten map, i.e., — see eq. (3.4) —

$$\Omega(x) = g(x) + \frac{h}{4}\omega^{mn}\{\tilde{a}_m, \partial_n g\}(x) + O(h^2).$$

Now, since $\tilde{\delta}_\xi$ and $\tilde{\delta}_\eta$ are infinitesimal variations, their commutator $[\tilde{\delta}_\xi, \tilde{\delta}_\eta]$ acts as a derivation on polynomials of the ordinary fields and their space-time derivatives. Then

$$[\tilde{\delta}_\xi, \tilde{\delta}_\eta]\mathcal{U}[\tilde{a}_m, u] = \mathcal{U}[(1 + [\tilde{\delta}_\xi, \tilde{\delta}_\eta])\tilde{a}_m, (1 + [\tilde{\delta}_\xi, \tilde{\delta}_\eta])u] - \mathcal{U}[\tilde{a}_m, u] + \text{higher orders},$$

where $\mathcal{U}[\tilde{a}_m, u]$ is the formal power series expansion that implements the Seiberg-Witten map. Taking into account eq. (3.6), one concludes that

$$[\hat{\delta}_\xi, \hat{\delta}_\eta]\mathcal{U}[\tilde{a}_m, u] = [\tilde{\delta}_\xi, \tilde{\delta}_\eta]\mathcal{U}[\tilde{a}_m, u] = \mathcal{U}[(1 + [\tilde{\delta}_\xi, \tilde{\delta}_\eta])\tilde{a}_m, (1 + [\tilde{\delta}_\xi, \tilde{\delta}_\eta])u] - \mathcal{U}[\tilde{a}_m, u] + \text{higher orders}. \quad (3.9)$$

On the other hand, eq. (3.5) leads to

$$\begin{aligned} [\hat{\delta}_\xi, \hat{\delta}_\eta]\mathcal{U}[\tilde{a}_m, u] &= (P + \delta_{\Omega(x)}^{(ncgauge)})\mathcal{U}[\tilde{a}_m, u] = (P + \delta_{g(x)}^{(gauge)})\mathcal{U}[\tilde{a}_m, u] \\ &= \mathcal{U}[(1 + P + \delta_{g(x)}^{(gauge)})\tilde{a}_m, (1 + P + \delta_{g(x)}^{(gauge)})u] - \mathcal{U}[\tilde{a}_m, u] + \text{higher orders}, \end{aligned} \quad (3.10)$$

upon using the fact that by definition of the Seiberg-Witten map we have $\delta_{\Omega(x)}^{(ncgauge)}\mathcal{U}[\tilde{a}_m, u] = \delta_{g(x)}^{(gauge)}\mathcal{U}[\tilde{a}_m, u]$. Finally, eqs. (3.9) and (3.10) imply that

$$\mathcal{U}[(1 + [\tilde{\delta}_\xi, \tilde{\delta}_\eta])\tilde{a}_m, (1 + [\tilde{\delta}_\xi, \tilde{\delta}_\eta])u] = \mathcal{U}[(1 + P + \delta_{g(x)}^{(gauge)})\tilde{a}_m, (1 + P + \delta_{g(x)}^{(gauge)})u],$$

which in turn yields eq. (3.8). Let us stress that the two key facts we have taken advantage of to obtain eq. (3.8) are that our noncommutative fields carry a representation of the supersymmetry algebra and that the Seiberg-Witten map turns (ordinary) gauge transformations of the ordinary fields into (noncommutative) gauge transformations of the noncommutative fields. Our proof of eq. (3.8) is valid to all orders in powers of $h\omega^{mn}$ and for any type of U(N) Seiberg-Witten map provided $\tilde{\delta}_\epsilon u(x)$ exists.

To close the current section let us remark that having a nonlinear realisation of the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions as furnished by the transformations in eq. (3.7) is in keeping with the duality that seems to establish the standard Seiberg-Witten map — supplemented with a field redefinition — between two supersymmetric DBI actions in four dimensions, namely, the noncommutative U(1) supersymmetric DBI action and the ordinary U(1) supersymmetric DBI action in the presence of a background field B_{mn} . Indeed, we show in appendix B that a given field redefinition of the Seiberg-Witten map in eq. (3.2) turns, for small B_{mn} and up to order 4 in the susy field strength, the ordinary U(1) supersymmetric DBI action for a background field B_{mn} in four dimensions into the leading contribution to the noncommutative U(1) supersymmetric DBI action; the latter being the action of noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills theory. Now, in four dimensions, the gauge supermultiplet of the ordinary U(1) supersymmetric DBI theory in a background field B_{mn} , as formulated in ref. [1], carries a nonlinear realisation of the $\mathcal{N} = 1$ supersymmetry algebra which is an unbroken symmetry of the corresponding DBI action. This nonlinear realisation of the supersymmetry algebra is [1] a B_{mn} -dependent linear combination of the extensions to the case of nonvanishing B_{mn} of the linear (unbroken) and the nonlinear (broken) supersymmetry transformations that leave invariant the DBI action for $B_{mn} = 0$ in four dimensions.

4. Only one dual ordinary theory

In section 2, we constructed an ordinary U(1) gauge theory whose fields carry a linear realisation of $\mathcal{N} = 1$ supersymmetry in four dimensions and is dual under the Seiberg-Witten map for superfields to noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills. The Seiberg-Witten map that connects these ordinary and noncommutative supersymmetric gauge theories is nonlocal — see eqs. (2.9) — but its nonlocal contributions are mere gauge artifacts. In section 3, we used the standard — local — Seiberg-Witten map in the Wess-Zumino gauge to construct an ordinary dual of noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills, the ordinary fields of this ordinary dual carrying a nonlinear realisation of the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions. The standard Seiberg-Witten map giving the latter ordinary dual of noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills is given in eq. (3.2). The purpose of the current section is to show, at first order in $h\omega^{mn}$, that the ordinary duals of noncommutative U(1) $\mathcal{N} = 1$ superYang-Mills that we have constructed in sections 2 and 3 are not different ordinary U(1) supersymmetric gauge theories but, indeed, the same ordinary theory each time formulated in terms of a different set of field variables: one set of fields represents the $\mathcal{N} = 1$ supersymmetry algebra linearly and the other set nonlinearly. Before we show this, we must change, as usual, the normalisation of the noncommutative, (A_m, Λ_α, D) , and ordinary, $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$, gauge supermultiplets of section 3 so that their gauge transformations have the same normalisation as the gauge transformations for components derived from the superfield gauge transformations used in section 2. The normalisation change in question is the following: $(A_m, \Lambda_\alpha, D) \rightarrow (\frac{1}{2}A_m, \Lambda_\alpha, D)$ and $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d}) \rightarrow (\frac{1}{2}\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$. This change of normalisation turns the the Seiberg-Witten

map in eq. (3.2) into the following Seiberg-Witten map:

$$\begin{aligned}
 A_m[\tilde{a}_n] &= \tilde{a}_m + hA_m^{(1)st} + O(h^2), & A_m^{(1)st} &= \frac{1}{2}\omega^{nl}\left(\tilde{a}_n\partial_l\tilde{a}_m - \frac{1}{2}\tilde{a}_n\partial_m\tilde{a}_l\right), \\
 \Lambda_\alpha[\tilde{a}_m, \tilde{\lambda}_\alpha] &= \tilde{\lambda}_\alpha + h\Lambda_\alpha^{(1)st} + O(h^2), & \Lambda_\alpha^{(1)st} &= \frac{1}{2}\omega^{mn}\tilde{a}_m\partial_n\tilde{\lambda}_\alpha, \\
 D[\tilde{a}_m, \tilde{d}] &= \tilde{d} + hD^{(1)st} + O(h^2), & D^{(1)st} &= \frac{1}{2}\omega^{mn}\tilde{a}_m\partial_n\tilde{d}.
 \end{aligned} \tag{4.1}$$

Let us next establish a map between the ordinary gauge supermultiplet (a_m, λ_α, d) that occurs in the map in eq. (2.14) and the ordinary gauge supermultiplet $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$ that is in the Seiberg-Witten map in eq. (4.1). We shall first remind the reader that the map between the noncommutative supermultiplet (A_m, Λ_α, D) and the ordinary supermultiplet (a_m, λ_α, d) defined by $V^{\text{WZ}}[a_m, \lambda_\alpha, d]$ in eq. (2.14) is obtained by gauge transforming to the Wess-Zumino gauge — see eqs. (2.10) to (2.14) — the Seiberg-Witten map defined by eqs. (2.3) and (2.9), when $x = 0$ and $\mathcal{X} = 0$ — recall that $\mathcal{X} = 0$ corresponds to an ordinary local field redefinition and therefore bears no physical consequences. Now, one may show that $A_{\dot{\beta}\gamma}^{(1)}, \Lambda_\rho^{(1)}$ and $D^{(1)}$ in eqs. (2.13) and (2.14) can be expressed as follows

$$\begin{aligned}
 A_{\dot{\beta}\gamma}^{(1)} &= A_{\dot{\beta}\gamma}^{(1)st} - 2\partial_{\dot{\beta}\gamma}\mathcal{Z} + \mathcal{A}_{\dot{\beta}\gamma}, \quad s\mathcal{A}_{\dot{\beta}\gamma} = 0, \\
 \Lambda_\rho^{(1)} &= \Lambda_\rho^{(1)st} + \mathcal{L}_\rho, \quad s\mathcal{L}_\rho = 0, \\
 D^{(1)} &= D^{(1)st} + \mathcal{D}, \quad s\mathcal{D} = 0,
 \end{aligned} \tag{4.2}$$

where $A_{\dot{\beta}\gamma}^{(1)st}$, $\Lambda_\rho^{(1)st}$ and $D^{(1)st}$ are obtained from the functions denoted with the same symbol in eq. (4.1) by replacing $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$ with (a_m, λ_α, d) . \mathcal{Z} and the BRS trivial pieces $\mathcal{A}_{\dot{\beta}\gamma}, \mathcal{L}$ and \mathcal{D} are displayed next:

$$\begin{aligned}
 \mathcal{Z} &= -\frac{1}{128}\omega^{\alpha\beta}\left(a_\alpha^{\dot{\alpha}} - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}\partial a\right)a_{\dot{\alpha}\beta} - \frac{1}{128}\omega^{\dot{\alpha}\dot{\beta}}\left(a_{\dot{\alpha}}^\alpha - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}\partial a\right)a_{\dot{\beta}\alpha}, \\
 \mathcal{A}_{\dot{\beta}\gamma} &= \frac{-1}{256}\omega^{\alpha\beta}\left[4\left(a_\alpha^{\dot{\alpha}} - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}\partial a\right)\partial_{\dot{\beta}\gamma}\left(a_{\dot{\alpha}\beta} - \frac{\partial_{\dot{\alpha}\beta}}{\square}\partial a\right) - 8\left(a_\alpha^{\dot{\alpha}} - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}\partial a\right)\partial_{\dot{\alpha}\beta}\left(a_{\dot{\beta}\gamma} - \frac{\partial_{\dot{\beta}\gamma}}{\square}\partial a\right) \right. \\
 &\quad \left. - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}d\frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\beta}\gamma}}{\square}d + 2i\frac{\partial_{\dot{\alpha}}^\alpha\partial_{\dot{\beta}\sigma}}{\square}\lambda^\sigma\frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\gamma}\dot{\sigma}}}{\square}\bar{\lambda}^{\dot{\sigma}}\right] + (c.c.)|_{\beta\leftrightarrow\gamma}, \\
 \mathcal{L}_\rho &= \frac{1}{128}\omega^{\alpha\beta}\left[4\left(a_\alpha^{\dot{\alpha}} - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}\partial a\right)\partial_{\dot{\alpha}\beta}\lambda_\rho + 2\frac{\partial_{\dot{\alpha}}^\alpha\partial_{\dot{\sigma}}^\rho}{\square}\lambda^\sigma\partial_{\dot{\alpha}\beta}\left(a_{\dot{\rho}\rho} - \frac{\partial_{\dot{\rho}\rho}}{\square}\partial a\right) + i\frac{\partial_{\dot{\alpha}}^\alpha\partial_{\dot{\sigma}}^\rho}{\square}\lambda^\sigma\frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\rho}\rho}}{\square}d\right] \\
 &\quad + \frac{1}{128}\omega^{\dot{\alpha}\dot{\beta}}\left[4\left(a_{\dot{\alpha}}^\alpha - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}\partial a\right)\partial_{\dot{\beta}\alpha}\lambda_\rho + 2\frac{\partial_{\dot{\alpha}}^\alpha\partial_{\dot{\sigma}}^\rho}{\square}\lambda^\sigma\partial_{\dot{\beta}\alpha}\left(a_{\dot{\rho}\rho} - \frac{\partial_{\dot{\rho}\rho}}{\square}\partial a\right) + i\frac{\partial_{\dot{\alpha}}^\alpha\partial_{\dot{\sigma}}^\rho}{\square}\lambda^\sigma\frac{\partial_{\dot{\beta}\alpha}\partial_{\dot{\rho}\rho}}{\square}d\right], \\
 \mathcal{D} &= \frac{1}{128}\omega^{\alpha\beta}\left[4\left(a_\alpha^{\dot{\alpha}} - \frac{\partial_{\dot{\alpha}}^\alpha}{\square}\partial a\right)\partial_{\dot{\alpha}\beta}d + 2\partial_{\dot{\alpha}}^\alpha\left(a^{\sigma\rho} - \frac{\partial^{\sigma\rho}}{\square}\partial a\right)\frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\rho}\sigma}}{\square}d + \frac{\partial_{\dot{\alpha}}^\alpha\partial_{\dot{\sigma}}^\rho}{\square}\lambda^\sigma\partial_{\dot{\alpha}\beta}\bar{\lambda}^{\dot{\rho}} + \right. \\
 &\quad \left. + \partial_{\dot{\alpha}}^\alpha\lambda^\sigma\frac{\partial_{\dot{\alpha}\beta}\partial_{\dot{\sigma}\rho}}{\square}\bar{\lambda}^{\dot{\rho}}\right] + (c.c.).
 \end{aligned} \tag{4.3}$$

We finally define the following maps between the ordinary gauge supermultiplets (a_m, λ_α, d) — linear — and $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$ — nonlinear:

$$\begin{aligned}\tilde{a}_m &= a_m - 2h\partial_m \mathcal{Z}[a] + h\mathcal{A}_m[a, \lambda, d] + O(h^2), \\ \tilde{\lambda}_\alpha &= \lambda + h\mathcal{L}_\alpha[a, \lambda, d] + O(h^2), \quad \tilde{d} = d + h\mathcal{D}[a, \lambda, d] + O(h^2),\end{aligned}\tag{4.4}$$

where \mathcal{Z} and the BRS-closed functions \mathcal{A}_m , \mathcal{L}_α and \mathcal{D} are given in eq. (4.3) — see also eq. (4.2).

Let us discuss some properties of the map in eq. (4.4). First, for infinitesimal U(1) transformations, it maps orbits of (a_m, λ_α, d) into orbits of $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$, and viceversa. Indeed, using eq. (4.4), one may show that

$$\tilde{s}_z(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d}) = s_z(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d}), \quad \tilde{z} = z + hs_z \mathcal{Z}[a_n],$$

where s_z denotes the U(1) BRS operator acting on (a_m, λ_α, d) : $s_z a_m = -2\partial_m z$, $s_z \lambda_\alpha = 0$ and $s_z d = 0$, and \tilde{s}_z stands for the U(1) BRS operator acting on $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$: $\tilde{s}_z \tilde{a}_m = -2\partial_m \tilde{z}$, $\tilde{s}_z \tilde{\lambda}_\alpha = 0$ and $\tilde{s}_z \tilde{d} = 0$. Secondly, the fact that under $\mathcal{N} = 1$ supersymmetry transformations the supermultiplet (a_m, λ_α, d) transforms as follows

$$\begin{aligned}\delta_\epsilon a_m &= \frac{1}{2}\epsilon\sigma_m \bar{\lambda} - \frac{1}{2}\bar{\epsilon}\sigma_m \lambda, & \delta_\epsilon \lambda_\alpha &= -\epsilon_\alpha d + i\epsilon_\gamma (\sigma^{mn})^\gamma{}_\alpha f_{mn}, \\ \delta_\epsilon d &= i\bar{\epsilon}\sigma^m \partial_m \lambda + i\epsilon\sigma^m \partial_m \bar{\lambda}, & f_{mn} &= \partial_m a_n - \partial_n a_m,\end{aligned}\tag{4.5}$$

and eq. (4.4), lead to

$$\tilde{\delta}_\epsilon(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d}) = (\tilde{\delta}_\epsilon + \tilde{s}_z)(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d}), \quad \tilde{z} = \text{Re}(ih\bar{\epsilon}\bar{\Psi}^{(1)}) + hs_{\text{Re}(ih\bar{\epsilon}\bar{\Psi}^{(1)})} \mathcal{Z}[a],$$

where $\tilde{\delta}_\epsilon(\tilde{a}, \tilde{\lambda}, \tilde{d})$ are the nonlinear supersymmetry transformations in eq. (3.7) for U(1) fields after the rescaling $\tilde{a}_m \rightarrow \frac{1}{2}\tilde{a}_m$, and $\bar{\Psi}^{(1)}$ and $\mathcal{Z}[a]$ are given in eqs. (2.13) and (4.3), respectively. Hence, modulo gauge transformations, the linear supersymmetry transformations — eq. (4.5) — of the gauge supermultiplet (a_m, λ_α, d) imply the nonlinear supersymmetry transformations of the gauge supermultiplet $\tilde{\delta}_\epsilon(\tilde{a}, \tilde{\lambda}, \tilde{d})$ as defined in eq. (3.7); and viceversa. Finally, if $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$ and (a_m, λ_α, d) satisfy eq. (4.4), then both gauge supermultiplets will have the same noncommutative supermultiplet image, (A_m, Λ_α, D) , under the corresponding maps in eqs. (2.14) and (4.1):

$$\begin{aligned}A_m &= a_m + hA_m^{(1)}[a_n \lambda_\alpha, d] + O(h^2) = \tilde{a}_m + hA_m^{(1)st}[\tilde{a}_n] + O(h^2), \\ \Lambda_\alpha &= \lambda_\alpha + h\Lambda_\alpha^{(1)}[a_n \lambda_\beta, d] + O(h^2) = \tilde{\lambda}_\alpha + h\Lambda_\alpha^{(1)st}[\tilde{a}_n, \tilde{\lambda}_\beta] + O(h^2), \\ D &= d + hD^{(1)}[a_n \lambda_\alpha, d] + O(h^2) = \tilde{d} + hD^{(1)st}[\tilde{a}_n, \tilde{d}] + O(h^2).\end{aligned}\tag{4.6}$$

Eq. (4.2) helps to show the previous set of equalities. We have thus shown that the supermultiplets $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$ and (a_m, λ_α, d) define, up to first order in $h\omega^{mn}$ the same U(1) ordinary supersymmetric gauge theory with no matter fields. Notice that eqs. (4.6) imply that the action in terms of $(\tilde{a}_m, \tilde{\lambda}_\alpha, \tilde{d})$ is equal to the action in terms of (a_m, λ_α, d) , if these gauge supermultiplets are related by eq. (4.4).

We have thus shown that the ordinary theories dual to noncommutative SYM found in sections 2 and 3 are not different theories but the same ordinary supersymmetric gauge theory formulated in each case in terms of a different set of field variables. The ordinary field variables introduced in section 2 carry a linearly realised $\mathcal{N} = 1$ supersymmetry and the set of ordinary fields of section 3 transforms nonlinearly under $\mathcal{N} = 1$ supersymmetry.

5. Summary and conclusions

In section 2, we have found, at first order in $h\omega^{mn}$, the most general solution to the Seiberg-Witten map equations for a noncommutative $U(1)$ vector superfield that is a polynomial in its ordinary counterpart, v , the chiral and antichiral projections of the latter, v_+ and v_- , and the susy covariant derivatives of them all; such polynomial being at most quadratic in v , v_+ and v_- . These Seiberg-Witten maps are nonlocal, but their nonlocal parts are gauge artifacts since they can be set to zero by choosing the supersymmetric Landau gauge. Furnished with this family of solutions to the $U(1)$ Seiberg-Witten map equations, we have obtained an ordinary dual under the Seiberg-Witten map of noncommutative SYM. This ordinary dual when formulated in terms of the ordinary fields considered in section 2 has linearly realised supersymmetry. In section 2, we have also shown by explicit computation that the standard Seiberg-Witten map of ref. [1] is never the $\theta\bar{\theta}$ component of a vector superfield which is a polynomial in the corresponding ordinary vector superfield and its susy covariant derivatives. In section 3, we have obtained the ordinary duals under the generalisation of the standard Seiberg-Witten map of ref. [1] of noncommutative $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry. These duals have been obtained by formulating the noncommutative theory in the Wess-Zumino gauge. The noncommutative fields of our noncommutative theory carry a linear realisation of the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions; however, as we have shown in section 3, their ordinary counterparts under the standard Seiberg-Witten map carry a nonlinear representation of the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions. Hence, the ordinary dual of our noncommutative supersymmetric theory supports a nonlinear realisation of the supersymmetry algebra when formulated in terms of the ordinary supermultiplets of section 3. We have seen that this is in line with the duality under the Seiberg-Witten map — see appendix B — between the noncommutative $U(1)$ supersymmetric DBI theory and the ordinary abelian supersymmetric DBI theory in a B_{mn} field in four dimensions. In section 4, we have shown that the ordinary duals of noncommutative SYM constructed in sections 2 and 3 by using completely different types of Seiberg-Witten map are not different ordinary supersymmetric gauge theories, but the same ordinary theory formulated, in each case, in terms of a different set of field variables: a set of field variables carries a linear representation of $\mathcal{N} = 1$ supersymmetry algebra in four dimensions and the other set carries a nonlinear representation of this algebra. We define, in section 4, the map that realises the change of field variables and study the properties of the map: it maps infinitesimal gauge orbits into infinitesimal gauge orbits and turns the linear realisation of $\mathcal{N} = 1$ supersymmetry in section 2 into the $h\omega^{mn}$ -dependent nonlinear realisation of the latter in section 3.

We believe that the results we have obtained in sections 2 and 4 for $U(1)$ can be extended to $U(N)$ groups in the fundamental, antifundamental, adjoint and bifundamental representations. However, to obtain explicit expressions such as the Seiberg-Witten map for superfields in eqs. (2.9) will be much harder since the r.h.s. in eq. (2.4) contains an infinite number of terms for nonabelian ordinary groups. We also believe that the results obtained in section 2 can be extended to any ordinary nonabelian gauge group in any representation, if one adopts the general philosophy behind the formalism put forward in refs. [24–26] for non-supersymmetric gauge theories: now the noncommutative vector superfields will be valued in the enveloping algebra of the Lie algebra of the ordinary gauge group. Section 3, however, will not hold, in general, for a given ordinary gauge group in a given representation, e.g., $SU(N)$ in the fundamental representation. Indeed, generally speaking $\tilde{\delta}_\epsilon \tilde{a}_m$ as defined in eq. (3.7) is not valued in the Lie algebra of the gauge group, so it is not, in general, a variation of an ordinary gauge field. It so happens that for arbitrary gauge groups in arbitrary representations, if the enveloping-algebra-valued noncommutative fields of the gauge triplet (A_m, Λ_α, D) are defined in terms of ordinary fields by means of the standard Seiberg-Witten map, the linear supersymmetry transformations in eq. (3.1) are not given rise to by variations of the ordinary fields. In view of the important results — see refs. [27, 28] — achieved within the enveloping-algebra formalism of refs. [24, 25] and [26], it is worth exploring how to construct supersymmetric versions of the models in refs. [27] and [28]. Perhaps, one should look for $h\omega^{mn}$ -dependent nonlinear realisations of supersymmetry carried by ordinary fields that yield upon using the standard Seiberg-Witten map noncommutative fields that also carry an $h\omega^{mn}$ -dependent nonlinear realisation of supersymmetry. Let us notice that we cannot start with an ordinary gauge supermultiplet having standard linear supersymmetry transformations and then apply the standard Seiberg-Witten map to define the noncommutative fields, since, as we show in appendix C, the ordinary action dual to the action of noncommutative $U(1)$ gauge theory cannot be made supersymmetric under those linear supersymmetry transformations by adding local terms which are polynomials in $h\omega^{mn}$. Finally, perhaps, to generalise the formalism of refs. [24, 25] and [26] so as to include supersymmetry, one should use the ideas and techniques in ref. [47].

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A. Superspace conventions

Our superspace conventions are those of ref. [36]. The superspace coordinates are given by $x^m, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}$, with $\bar{\theta}_{\dot{\alpha}} = \theta_{\dot{\alpha}}^*$. We denote space-time indices with latin letters and spinor

indices with greek letters. Spinor indices are raised with and lowered with $\epsilon^{\alpha\beta}, \epsilon_{\alpha\beta}, \epsilon^{\dot{\alpha}\dot{\beta}}, \epsilon_{\dot{\alpha}\dot{\beta}}$ such that $\epsilon^{12} = 1 = \epsilon_{12} = -\epsilon^{\dot{1}\dot{2}} = -\epsilon_{\dot{1}\dot{2}}$ and $\epsilon_{\alpha\beta}^* = \epsilon_{\dot{\beta}\dot{\alpha}}$. Contractions will be denoted as $\epsilon\eta \equiv \epsilon^\alpha\eta_\alpha, \bar{\epsilon}\bar{\eta} \equiv \bar{\epsilon}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}}$. For the sigma matrices we have

$$\begin{aligned} (\sigma^m)^{\alpha\dot{\alpha}} &= (1, \vec{\sigma}), & (\bar{\sigma}^m)_{\dot{\alpha}\alpha} &= (-1, \vec{\sigma}), \\ (\sigma^{mn})^\alpha{}_\beta &= \frac{1}{2}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)^\alpha{}_\beta, & (\bar{\sigma}^{mn})_{\dot{\alpha}}{}^{\dot{\beta}} &= \frac{1}{2}(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m)_{\dot{\alpha}}{}^{\dot{\beta}}. \end{aligned}$$

Superfields are functions over the superspace. We denote noncommutative superfields with capital letters and ordinary superfields with lower-case letters. An ordinary superfield χ transforms under supersymmetry as

$$\delta_\epsilon\chi(x, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = (-\epsilon Q - \bar{\epsilon}\bar{Q})\chi(x, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}),$$

and identically for a noncommutative superfield Ξ . The generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ satisfy the supersymmetry algebra $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\bar{\sigma}_{\dot{\alpha}\alpha}^m \partial_m$; explicitly

$$Q_\alpha = \partial_\alpha + i\bar{\theta}^{\dot{\alpha}}(\bar{\sigma}^m)_{\dot{\alpha}\alpha} \partial_m, \quad \bar{Q}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i(\bar{\sigma}^m)_{\dot{\alpha}\alpha} \theta^\alpha \partial_m.$$

The supersymmetric covariant derivatives $D_\alpha, \bar{D}_{\dot{\alpha}}$, which satisfy $\{D_\alpha, Q_\beta\} = 0 = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}$ and $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\bar{\sigma}_{\dot{\alpha}\alpha}^m \partial_m$, are

$$D_\alpha = \partial_\alpha - i\bar{\theta}^{\dot{\alpha}}(\bar{\sigma}^m)_{\dot{\alpha}\alpha} \partial_m, \quad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - i(\bar{\sigma}^m)_{\dot{\alpha}\alpha} \theta^\alpha \partial_m.$$

We consider the following component expansion of a real superfield v :

$$\begin{aligned} v(x, \theta, \bar{\theta}) &= c(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}(x) + \frac{1}{2}\theta^2 f(x) + \frac{1}{2}\bar{\theta}^2 \bar{f}(x) + \theta^\alpha \bar{\sigma}_{\dot{\beta}\alpha}^m \bar{\theta}^{\dot{\beta}} a_m + \frac{1}{2}\theta^2 \bar{\theta}^{\dot{\alpha}} \lambda'_{\dot{\alpha}} \\ &\quad + \frac{1}{2}\bar{\theta}^2 \theta^\alpha \lambda'_\alpha + \frac{1}{4}\bar{\theta}^2 \theta^2 d', \\ \lambda'_\alpha &\equiv \lambda_\alpha - i\bar{\sigma}_{\dot{\beta}\alpha}^m \partial_m \bar{\psi}^{\dot{\beta}}, \quad d' \equiv d + \square c, \end{aligned}$$

and similarly for a noncommutative real superfield V .

B. Duality between noncommutative and ordinary supersymmetric U(1) DBI theories

The aim of this appendix is to show the equivalence of the effective supersymmetric DBI actions for open strings ending on D-branes obtained, on the one hand, in noncommutative space-time, and on the other, in ordinary space-time but in the presence of a constant background B_{mn} . The first type of DBI actions have a linearly realised supersymmetry in terms of the noncommutative fields, while the ordinary DBI actions with a B_{mn} background are invariant under non-linear supersymmetry transformations. The equivalence is provided by the Seiberg-Witten maps; this provides a natural understanding of the fact that ordinary fields in local SW maps always seem to transform non-linearly under supersymmetry.

In the non-supersymmetric U(1) case, the equivalence was first noted by Seiberg and Witten [1], and it was shown to be exact. In the supersymmetric case, both the ordinary

and noncommutative actions are known — see refs. [48, 49] — but their possible equivalence has not been studied. Here we will show the equivalence in the limit of $h\omega \rightarrow 0$ and for small values of the fields. We choose the $h\omega^{mn} \rightarrow 0$ and not the Seiberg-Witten limit $\alpha' \rightarrow 0$ because in the supersymmetric case the $\alpha' \rightarrow 0$ limit requires a complicated reexpansion of the action, while the $h\omega^{mn} \rightarrow 0$ limit is compatible with a perturbative definition of the DBI actions in terms of an expansion in the number of fields. Our aim is to show the equivalence of the DBI actions at first order in h and up to products of three ordinary fields. The noncommutative DBI lagrangian, which we shall denote as $\hat{\mathcal{L}}_{\text{DBI}}$, is a functional of the noncommutative supersymmetric field strengths $\hat{W}_\alpha = -\frac{1}{4}\bar{D}^2(e_\star^{-V} \star D_\alpha e_\star^V)$. It is given by a sum of terms with even powers of $\hat{W}^2, \hat{\bar{W}}^2$ [49], so that it involves sums of products of an even number of component fields. We want to expand this action in terms of ordinary fields at first order in h using the standard SW maps of eq. (4.1).

It can be easily seen that in order to compute the contributions with products of three ordinary fields and less, we only need $\hat{\mathcal{L}}_{\text{DBI}}$ up to $O(\hat{W}^2)$. Thus, following [49] — see [1] for the normalisation — for a D3 brane we get,

$$\hat{\mathcal{L}}_{\text{DBI}} = \frac{1}{2\pi G_s} \left(\frac{1}{16} \int d^2\theta \hat{W}^2 + \frac{1}{16} \int d^2\bar{\theta} \hat{\bar{W}}^2 \right) + O(\hat{W}^4), \quad (\text{B.1})$$

where G_s is the noncommutative string coupling constant. In the component field expansion of \hat{W} one must use the noncommutative space-time metric G .

On the other hand, concerning the ordinary DBI action in the presence of the background field B_{mn} — which we shall denote as \mathcal{L}_{DBI} — it is constructed from the action with $B_{mn} = 0$ by making the substitution $f_{mn} \rightarrow f_{mn} - 2B_{mn}$ — the differences with the conventions in [1] are due to our choice of the component field expansion of the superfield v . The action at $B_{mn} = 0$ is given by an expansion involving even powers of W^2 , where $W_\alpha = -\frac{1}{4}D^2 D_\alpha v$ is the ordinary supersymmetric field-strength, so that to get the terms with three fields after the substitution $f_{mn} \rightarrow f_{mn} - 2B_{mn}$ we need the terms of $\mathcal{L}_{\text{DBI}}^{B=0}$ up to $O(W^4)$. These are given, adapting the result in [48] to our conventions, by the following expression

$$\mathcal{L}_{\text{DBI}} = \frac{1}{2\pi g_s} \left(\frac{1}{16} \int d^2\theta W^2 + \frac{1}{16} \int d^2\bar{\theta} \bar{W}^2 + \frac{(2\pi\alpha')^2}{128} \int d^2\theta d^2\bar{\theta} W^2 \bar{W}^2 + O(W^6) \right) \Big|_{f \rightarrow f - 2B}. \quad (\text{B.2})$$

g_s is the ordinary string coupling constant, and the ordinary metric g must be used in the component field expansion of W .

In order to relate both of the actions (B.1) and (B.2) in the limit of small $h\omega$, we need the results from [1] that follow

$$\frac{1}{G_s} = \frac{1}{g_s} + O(h^2), \quad G^{mn} = g^{mn} + O(h^2), \quad B = \frac{-1}{(2\pi\alpha')^2} g^{-1} h\omega g^{-1} + O(h^2). \quad (\text{B.3})$$

For simplicity we can take both G and g as the Minkowski metric. We must expand both of the actions (B.1) and (B.2) in terms of the ordinary component fields and compare the results. Using the SW maps in (4.1), the noncommutative action $\hat{\mathcal{L}}_{\text{DBI}}$ is given by

$$\hat{\mathcal{L}}_{\text{DBI}} = \frac{1}{2\pi g_s} \left[-\frac{1}{16} f_{mn} f^{mn} + \frac{i}{16} \bar{\lambda} \bar{\sigma}^m \partial_m \lambda + \frac{1}{32} d^2 - \frac{h}{64} \omega^{kl} f_{kl} f_{ij} f^{ij} + \frac{h}{16} \omega^{kl} f_{ik} f_{jl} f^{ij} \right]$$

$$\begin{aligned}
 & + \frac{i\hbar}{128} \omega^{kl} f_{kl} (\bar{\lambda} \bar{\sigma}^m \partial_m \lambda - \partial_m \bar{\lambda} \bar{\sigma}^m \lambda) + \frac{i\hbar}{64} \omega^{kl} f_{mk} (\bar{\lambda} \bar{\sigma}^m \partial_l \lambda - \partial_l \bar{\lambda} \bar{\sigma}^m \lambda) + \frac{h}{128} \omega^{kl} f_{kl} d^2 \Big] \\
 & \hspace{15em} \text{(B.4)}
 \end{aligned}$$

+ $O(4 \text{ fields}) + O(\hbar^2) + \text{total derivative}$.

The ordinary \mathcal{L}_{DBI} action in eq. (B.2) has the following component expansion, after using the relation between ω and B in eq. (B.3):

$$\begin{aligned}
 \mathcal{L}_{\text{DBI}} = \frac{1}{2\pi g_s} \Bigg[& -\frac{1}{16} f_{mn} f^{mn} + \frac{i}{16} \bar{\lambda} \bar{\sigma}^m \partial_m \lambda + \frac{1}{32} d^2 - \frac{h}{64} \omega^{kl} f_{kl} f_{ij} f^{ij} + \frac{h}{16} \omega^{kl} f_{ik} f_{jl} f^{ij} \\
 & - \frac{i\hbar}{256} \omega^{kl} f_{kl} (\bar{\lambda} \bar{\sigma}^m \partial_m \lambda - \partial_m \bar{\lambda} \bar{\sigma}^m \lambda) + \frac{h}{256} \tilde{\omega}^{kl} f_{kl} \partial_m (\bar{\lambda} \bar{\sigma}^m \lambda) \\
 & + \frac{i\hbar}{128} \omega^{kl} f_{mk} (\bar{\lambda} \bar{\sigma}_l \partial^m \lambda - \partial^m \bar{\lambda} \bar{\sigma}_l \lambda) + \frac{i\hbar}{128} \omega^{kl} f_{mk} (\bar{\lambda} \bar{\sigma}^m \partial_l \lambda - \partial_l \bar{\lambda} \bar{\sigma}^m \lambda) \\
 & - \frac{h}{128} \epsilon^{lmqt} \omega_{kl} f_m^k \partial_q (\bar{\lambda} \bar{\sigma}_t \lambda) - \frac{h}{256} \omega^{kl} d \partial_l (\bar{\lambda} \bar{\sigma}_k \lambda) + \frac{i\hbar}{256} \tilde{\omega}^{kl} d (\partial_k \bar{\lambda} \bar{\sigma}_l \lambda - \bar{\lambda} \bar{\sigma}_l \partial_k \lambda) \\
 & - \frac{h}{128} \omega^{kl} f_{kl} d^2 \Big] + O(4 \text{ fields}) + O(\hbar^2) + \text{total derivative},
 \end{aligned}$$

where we have defined $\tilde{\omega}^{kl} \equiv \frac{1}{2} \epsilon^{klmn} \omega_{mn}$. At first sight, it is clear that the terms involving f_{mn} alone coincide, as is known from previous results concerning non-supersymmetric theories. Still, the rest of the terms do not seem to match. However, we must still note that the SW maps are not uniquely defined, since they have an ambiguity given, in the $U(1)$ case, by field redefinitions. Hence, we should check whether redefining the fields in the lagrangian \mathcal{L}_{DBI} we can exactly match $\hat{\mathcal{L}}_{\text{DBI}}$ of eq. (B.4). The answer turns out to be positive in a non-trivial way. Indeed, it can be seen after some work that the following field redefinitions

$$\begin{aligned}
 \delta a_m &= \frac{h}{16} \tilde{\omega}_m^{\ n} \bar{\lambda} \bar{\sigma}_n \lambda, \\
 \delta \lambda_\alpha &= -\frac{3i\hbar}{16} \tilde{\omega}^{kl} f_{kl} \lambda_\alpha + \frac{3h}{16} \omega^{kl} f_{kl} \lambda_\alpha + \frac{h}{8} \omega^{kl} f_{mk}^m (\sigma_{lm})^{\alpha\beta} \lambda_\beta + \frac{i\hbar}{4} \tilde{\omega}^{kl} f_{mk}^m (\sigma_{lm})^{\alpha\beta} \lambda_\beta, \\
 \delta d &= \frac{h}{4} \omega^{kl} f_{kl} d - \frac{h}{16} \omega^{kl} \partial_k (\bar{\lambda} \bar{\sigma}_l \lambda) - \frac{i\hbar}{16} \tilde{\omega}^{kl} (\partial_k \bar{\lambda} \bar{\sigma}_l \lambda - \bar{\lambda} \bar{\sigma}_l \partial_k \lambda).
 \end{aligned}$$

turn \mathcal{L}_{DBI} into $\hat{\mathcal{L}}_{\text{DBI}}$, modulo total derivatives and working at order h and with terms involving products of up to three component fields. This is not trivial since even when considering the previous field redefinitions with arbitrary coefficients for the different terms, one cannot generate in the action \mathcal{L}_{DBI} the terms appearing in $\hat{\mathcal{L}}_{\text{DBI}}$ with arbitrary coefficients. This shows that both DBI actions are in fact equivalent at least in the limit of small $h\omega^{mn}$ and small values of the fields, and this equivalence is provided by the Seiberg-Witten map in eq. (4.1) supplemented with the previous field redefinitions. I.e., the modified Seiberg-Witten maps that follow,

$$\begin{aligned}
 A_m &= a_m + \frac{h}{2} \omega^{kl} \left(a_k \partial_l a_m - \frac{1}{2} a_k \partial_m a_l \right) - \frac{h}{16} \tilde{\omega}_m^{\ n} \bar{\lambda} \bar{\sigma}_n \lambda + O(\hbar^2), \\
 \Lambda &= \lambda + \frac{h}{2} \omega^{kl} a_k \partial_l \lambda + \frac{3i\hbar}{16} \tilde{\omega}^{kl} f_{kl} \lambda_\alpha - \frac{3h}{16} \omega^{kl} f_{kl} \lambda_\alpha - \frac{h}{8} \omega^{kl} f_{mk}^m (\sigma_{lm})^{\alpha\beta} \lambda_\beta - \frac{i\hbar}{4} \tilde{\omega}^{kl} f_{mk}^m (\sigma_{lm})^{\alpha\beta} \lambda_\beta
 \end{aligned}$$

$$\begin{aligned}
 & + O(h^2), \\
 D = & d + \frac{h}{2}\omega^{kl}a_k\partial_l d - \frac{h}{4}\omega^{kl}f_{kl}d + \frac{h}{16}\omega^{kl}\partial_k(\bar{\lambda}\bar{\sigma}_l\lambda) + \frac{ih}{16}\tilde{\omega}^{kl}(\partial_k\bar{\lambda}\bar{\sigma}_l\lambda - \bar{\lambda}\bar{\sigma}_l\partial_k\lambda) + O(h^2).
 \end{aligned}$$

map $\hat{\mathcal{L}}_{\text{DBI}}$ of eq. (B.1) into the action \mathcal{L}_{DBI} of eq. (B.2).

It is worth noting that, in the pure bosonic case, there is no need to consider field redefinitions; in fact the equivalence of the pure bosonic parts of $\hat{\mathcal{L}}_{\text{DBI}}$ and \mathcal{L}_{DBI} was shown to be exact without having to use field redefinitions. This is due to the fact that, at least at order h and possibly beyond, the pure bosonic field redefinitions only modify the bosonic lagrangian with pure derivative terms, so that their effect can be neglected.

C. Is there a local linear supersymmetric completion of the bosonic Yang-Mills action expanded with the standard SW map?

In section 1 it was shown that the standard SW map can never be embedded into a superfield. Furthermore, we have seen that when considering local SW maps in components, the ordinary fields transform in a non-linear representation of the supersymmetry algebra. In all these cases, it was assumed that supersymmetry was linearly realised on the side of the noncommutative fields. However, there is still the possibility of the ordinary fields being in a linear representation of supersymmetry and the noncommutative ones in a non-linear one. We can thus start assuming a linear representation of supersymmetry on the WZ gauge component fields a_m, λ, d , i.e., they should transform as in eq. (4.5). With this point of view, the transformation properties of the noncommutative fields are unknown and so is the action in terms of noncommutative fields. Nevertheless, we know its pure bosonic part, which is the noncommutative Yang-Mills expanded with the SW map. Assuming further that the standard SW map (2.5) is valid for the A_m component, we have that the bosonic part of the action is given by

$$\begin{aligned}
 S_{\text{bosonic}} = & -\frac{1}{16}\int d^4x F_{mn} \star F^{mn} = \\
 & -\frac{1}{16}\int d^4x f_{mn}f^{mn} - \frac{h}{64}\int d^4x \omega^{ab}f_{ab}f_{mn}f^{mn} + \frac{h}{16}\int d^4x \omega^{ab}f_{ma}f_{nb}f^{mn} + O(h^2),
 \end{aligned} \tag{C.1}$$

where the awkward normalisation factors are due to our unconventional definitions of the component fields A_m, a_m . What needs to be checked is whether there is any local, Poincaré and gauge invariant completion of the action (C.1) involving the WZ component fields a_m, λ, d which is invariant under the supersymmetric transformations of eq. (4.5). Since the order $O(h^0)$ part is known to have a supersymmetric completion, it suffices to check the $O(h)$ part. To do so we consider all the possible independent — modulo integration by parts — gauge invariant monomials which are of order one in ω^{mn} , constructed from the fields a_m, λ, d and spacetime derivatives, which include at least one superpartner field λ, d . They are shown next:

$$\begin{aligned}
 t_1 = & \omega^{mn}f_{mr}f_n{}^r D, & t_2 = & \omega^{mn}f_{mn}d^2, & t_3 = & \omega^{mn}\partial_r f_{mn}\lambda\sigma^r\bar{\lambda}, \\
 t_4 = & \omega^{mn}\partial^r f_{mr}\lambda\sigma_n\bar{\lambda}, & t_5 = & \omega^{mn}\partial_n d\lambda\sigma_m\bar{\lambda}, & t_6 = & \omega^{mn}f_{mn}\square d,
 \end{aligned}$$

$$\begin{aligned}
t_7 &= \text{Im } \omega^{mn} f_{mn} \lambda \not{\partial} \bar{\lambda}, & t_8 &= \text{Im } \omega^{mn} f_{mr} \lambda \sigma_n \partial^r \bar{\lambda}, & t_9 &= \text{Im } \omega^{mn} f_{mr} \lambda \sigma^r \partial_n \bar{\lambda}, \\
t_{10} &= \text{Im } \omega^{mn} d\lambda \sigma_m \partial_n \bar{\lambda}, & t_{11} &= \text{Im } \omega^{mn} \square \partial_m \lambda \sigma_n \bar{\lambda}.
\end{aligned}$$

“Im” denotes imaginary part. By solving

$$\widehat{\delta}_\epsilon \left[S^{\text{bosonic}} + h \int d^4x \sum_i \alpha_i t_i \right] = 0$$

expanding the l.h.s. in integrals of independent monomials, one readily finds that there is no solution to the previous equation. This can be seen for example by considering just the terms of the type $ff\lambda, ff\bar{\lambda}$, which are the only ones generated from the supersymmetric variation of the fff terms of the bosonic action, as is clear from eq. (4.5).

Thus, the noncommutative Yang-Mills action expanded with the standard SW map has no completion invariant under the linear supersymmetry from eq. (4.5).

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